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New Exact Solutions of the Yang–Mills Field Equations

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In this paper new exact solutions of the Yang-Mills SU(2) gauge field equations are obtained using the Carmeli-Charach-Kaye null-tetrad formalism. The solutions are classified and briefly discussed.

Classical gauge theories play an important role in particle physics and thus there is a considerable interest in obtaining exact solutions of the Yang-Mills field equations [see, e.g., Sarkar and Raychaudhuri (1982), Teh et al. (1982), Mathelitsch et al. (1982), Oh (1982a, b), Brown et al. (1982), O'Raifeartaigh et al. (1982), Elizalde (1983), Brown (1982), and Chang (1984); for a review see Actor (1979)]. It is believed that classical solutions give a better understanding and a deeper insight to the quantum theory.

In this paper we obtain new exact solutions of the Yang-Mills $SU(2)$ gauge field equations using classification scheme (Castillejo et al., 1979) in combination with the Carmeli-Charach-Kaye null-tetrad method (Carmeli et al., 1978; Carmeli, 1982). Consequently, a simplified form of the Yang-Mills fields and equations is derived. The use of the null-tetrad method in gravitation is well known in general relativity theory (Kinnersley, 1969). The extension to non-Abelian gauge theory of the null-tetrad method was carried out leading to simplification and new solutions (Carmeli, 1977; Altamirano and Villarroel, 1981). On the other hand, new solutions were also obtained (Castillejo and Kugler, 1981) using classification schemes, and more recently using the combination of both the classification scheme and the null-tetrad formlism (Carmeli and Huleihil, 1983).

We will look for solutions of the Yang-Mills field equations (Yang and Mills, 1954) that belong to class C_1, which have the following form

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(Castillejo et al., 1979):

$$A_i^k = \begin{pmatrix} a+ic & 0 & 0 \\ 0 & b+id & 0 \\ 0 & 0 & b+id \end{pmatrix} \tag{1}$$

Here $A_i^k = E_i^k + iB_i^k$, where E_i^k and B_i^k are the electric and the magnetic parts of the Yang–Mills fields, respectively. The quantities a, b, c, and d are real functions of the coordinates, and the lower case indices i, k take the values 1, 2, 3.

In the null-tetrad version, the ordinary fields and potentials of the Yang–Mills theory are given by the symmetric spinor χ_{abk} and the isotriplet $b_{ac'k}$, which are defined by

$$\chi_{abk} = \tfrac{1}{2}\varepsilon^{c'd'}\sigma^{\mu}_{ac'}\sigma^{\nu}_{bd'}f_{\mu\nu k} \tag{2}$$

$$b_{ac'k} = \sigma^{\mu}_{ac'}b_{\mu k} \tag{3}$$

Here $f_{\mu\nu k}$ and $b_{\mu k}$ are the ordinary Yang–Mills fields and potentials, $\sigma^{\mu}_{ac'}$ is a tetrad of null vectors, $\varepsilon^{c'd'}$ is the antisymmetric spinor given by

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

The indices a, b and c', d' are dyad indices taking the values 0, 1 and 0′, 1′, respectively, a bar indicates complex conjugation, and μ, ν are space-time indices taking the values 0, 1, 2, 3.

For the class of fields we are concerned with one finds, using equations (1) and (2),

$$\chi_{00k} = \frac{-1}{\sqrt{2}}[(a+ic)\mathscr{D}\alpha_k + (b+id)\mathscr{D}\beta_k] \tag{5}$$

$$\chi_{01k} = \tfrac{1}{2}[(a+ic)\alpha_k + (b+id)\beta_k] \tag{6}$$

$$\chi_{11k} = \frac{1}{2\sqrt{2}}[(a+ic)\bar{\mathscr{D}}\alpha_k + (b+id)\bar{\mathscr{D}}\beta_k] \tag{7}$$

where

$$\mathscr{D} = \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} \tag{8}$$

$$\alpha_k = (n_1, 0, 0) \tag{9}$$

$$\beta_k = (0, n_2, n_3) \tag{10}$$

and n_k is the unit vector given by

$$n_k = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{11}$$

The coordinates are defined by $x^0 = u$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, where $u = t - r$ is a retarded time coordinate, and the speed of light is taken as unity.

Substituting equations (5)-(7) in the Yang–Mills field equations (see Carmeli et al., 1978, and Carmeli, 1982), we get

$$b_{00'2}n_2 + \sqrt{2}b_{10'2}\mathscr{D}n_2 = b_{00'3}n_3 + \sqrt{2}b_{10'3}\mathscr{D}n_3 \tag{12}$$

$$-b_{00'2}\bar{\mathscr{D}}n_2 + \sqrt{2}b_{10'2}n_2 = -b_{00'3}\bar{\mathscr{D}}n_3 + \sqrt{2}b_{10'3}n_3 \tag{13}$$

$$\sqrt{2}b_{11'2}n_2 - b_{01'2}\bar{\mathscr{D}}n_2 = \sqrt{2}b_{11'3}n_3 - b_{01'3}\bar{\mathscr{D}}n_3 \tag{14}$$

$$\sqrt{2}b_{11'2}\mathscr{D}n_2 + b_{01'2}n_2 = \sqrt{2}b_{11'3}\mathscr{D}n_3 + b_{01'3}n_3 \tag{15}$$

One therefore has to solve the system of the four algebraic equations (12)-(15) of complex variables. We then obtain two independent solutions leading to the following two cases:

$$(1) \quad b_{00'2} = -2b_{11'2} = -n_3F \tag{16a}$$

$$b_{00'3} = -2b_{11'3} = n_2F \tag{16b}$$

$$b_{01'2} = \frac{-1}{\sqrt{2}}F\mathscr{D}n_3 \tag{16c}$$

$$b_{01'3} = \frac{1}{\sqrt{2}}F\mathscr{D}n_2 \tag{16d}$$

and

$$(2) \quad b_{00'2} = -2b_{11'2} = n_2H \tag{17a}$$

$$b_{00'3} = -2b_{11'3} = n_3H \tag{17b}$$

$$b_{01'2} = \frac{1}{\sqrt{2}}H\mathscr{D}n_2 \tag{17c}$$

$$b_{01'3} = \frac{1}{\sqrt{2}}H\mathscr{D}n_3 \tag{17d}$$

Here F and H are real functions of the coordinates u, r, θ, φ, and $b_{10'k}$ is the complex conjugate of $b_{01'k}$. It remains to find the functions F, H, a, b, c, d and the potentials $b_{00'1}$, $b_{01'1}$, and $b_{11'1}$.

Let us first consider the case (1). Substituting the potentials, equations (16), in the field equations and the relations between the fields and the potentials (Carmeli et al., 1978), then one finds that all the field variables are independent of u, and that

$$n_1(b_{00'1}-2b_{11'1})=-\sqrt{2}(b_{01'1}\bar{\mathscr{D}}n_1+b_{10'1}\mathscr{D}n_1) \tag{18a}$$

$$\frac{i}{grF}(\mathscr{D}n_1\bar{\mathscr{D}}F-\bar{\mathscr{D}}n_1\mathscr{D}F)=b_{00'1}-2b_{11'1} \tag{18b}$$

$$b=-\tfrac{1}{2}gF(b_{00'1}+2b_{11'1}) \tag{18c}$$

$$d=\frac{1}{n_1}\left[\frac{ig}{\sqrt{2}}F(b_{01'1}\bar{\mathscr{D}}n_1-b_{10'1}\mathscr{D}n_1)-\frac{\partial F}{\partial r}\right] \tag{18d}$$

$$c\mathscr{D}n_1=\frac{i}{2r}\mathscr{D}(b_{00'1}-2b_{11'1})-i\sqrt{2}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)b_{01'1}-gF^2\mathscr{D}n_1 \tag{18e}$$

$$a\mathscr{D}n_1=-\frac{1}{2r}\mathscr{D}(b_{00'1}+2b_{11'1}) \tag{18f}$$

$$an_1=-\frac{1}{2}\frac{\partial}{\partial r}(b_{00'1}+2b_{11'1}) \tag{18g}$$

where g is the coupling constant. From the last two equations one gets

$$\frac{n_1}{r}\mathscr{D}(b_{00'1}+2b_{11'1})=\mathscr{D}n_1\frac{\partial}{\partial r}(b_{00'1}+2b_{11'1}) \tag{19}$$

the solution of which is given by

$$b_{00'1}+2b_{11'1}=2K(x) \tag{20}$$

where $K(x)$ is a real function of the variable $x=rn_1=r\sin\theta\cos\varphi$. Using now equation (20) in equations (18c), (18f), and (18g) one obtains

$$a=-K' \tag{21}$$

$$b=-gFK \tag{22}$$

where a prime denotes a derivative with respect to the variable x. Inserting equations (21) and (22) in the field equations one finds that F is a function of x only, and hence

$$b_{00'1}=2b_{11'1}=K(x) \tag{23}$$

$$b_{01'1}=0 \tag{24}$$

$$c=-gF^2 \tag{25}$$

$$d=-F' \tag{26}$$

The functions F and K, furthermore, have to satisfy the following differential equations:

$$K'' - 2g^2F^2K = 0 \tag{27}$$

$$F'' - g^2F(F^2 - K^2) = 0 \tag{28}$$

In the same way we get the following results for case (2):

$$b_{00'1} = -2b_{11'1} = n_1 G(t) \tag{29}$$

$$b_{01'1} = \frac{1}{\sqrt{2}} \mathscr{D}_{n_1} G(t) \tag{30}$$

$$a = \dot{G} \tag{31}$$

$$b = \dot{H} \tag{32}$$

$$c = -gH^2 \tag{33}$$

$$d = -gHG \tag{34}$$

where a dot denotes differentiation with respect to the coordinate $t = r + u$, and G and H are real functions of t satisfying the differential equations

$$\ddot{G} + 2g^2H^2G = 0 \tag{35}$$

$$\ddot{H} + g^2H(H^2 + G^2) = 0 \tag{36}$$

To find the solution for each case we thus have to solve the system of two ordinary nonlinear differential equations (27), (28) and (35), (36). We now consider particular solutions for cases (1) and (2).

(1) Let us assume that the electric part of the Yang-Mills fields vanishes, and therefore $K = 0$. Then integrating equation (28) gives

$$F'^2 = (g^2F^4 \pm A^4)/2 \tag{37}$$

where $\pm A^4/2$ is an integration constant. The integration of the last equation is an elliptic integral of the first kind (Gradshteyn and Ryzik, 1965), and thus we get for $+A^4$,

$$F = \pm \frac{A}{\sqrt{g}} \left(\frac{1 - cn\chi}{1 + cn\chi} \right)^{1/2} \tag{38}$$

and for $-A^4$,

$$F = \pm \frac{A}{\sqrt{g}} \left(\frac{1 + cn^2\psi}{1 - cn^2\psi} \right)^{1/2} \tag{39a}$$

$$F = \frac{A}{\sqrt{g}\, cn\psi} \tag{39b}$$

where $\chi=(2g)^{1/2}A(x-x_0)$, $\psi=\chi/\sqrt{2}$ and $cn\chi$ is a Jacobian elliptic function.

(2) For case (2) we take $H=G$, then by integrating equation (36) we obtain

$$\dot{H}^2=B^4-g^2H^4 \tag{40}$$

where B^4 is an integration constant. The integration of equation (40) is an elliptic integral of the first kind, leading to

$$H=\pm\frac{B}{\sqrt{g}}\left(\frac{1-cn^2\tau}{1+cn^2\tau}\right)^{1/2} \tag{41a}$$

$$H=\frac{B}{\sqrt{g}}\,cn\tau \tag{41b}$$

where $\tau=(2g)^{1/2}B(t-t_0)$.

Finally, we give the Yang-Mills potentials and fields in their ordinary form in Cartesian coordinates.

Case 1:

$$b_\mu^k=(K\,\delta_1^k, 0, F\,\delta_3^k, F\,\delta_2^k), \tag{42}$$

$$\begin{aligned} f_{01}^k&=\delta_1^k K', & f_{02}^k&=g\,\delta_2^k FK, & f_{03}^k&=-g\,\delta_3^k FK \\ f_{12}^k&=-\delta_3^k F', & f_{23}^k&=g\,\delta_1^k F^2, & f_{31}^k&=\delta_2^k F' \end{aligned} \tag{43}$$

Case 2:

$$b_\mu^k=(0, G\,\delta_1^k, H\,\delta_2^k, H\,\delta_3^k) \tag{44}$$

$$\begin{aligned} f_{01}^k&=-\delta_1^k\dot{G}, & f_{02}^k&=-\delta_2^k\dot{H}, & f_{03}^k&=-\delta_3^k\dot{H} \\ f_{12}^k&=-g\,\delta_3^k GH, & f_{23}^k&=-g\,\delta_1^k H^2, & f_{31}^k&=-g\,\delta_2^k GH \end{aligned} \tag{45}$$

As mentioned, exact solutions of the Yang-Mills field equations are obtained. The fields of these solutions belong to class C_1 and therefore they have some symmetry which, together with the use of the null-tetrad formalism, simplified the calculation. Although we have no general solutions for the two systems of nonlinear differential equations (27), (28) and (35), (36), particular cases were solved. We hope that this work will stimulate to derive other solutions of the above mentioned class of fields, a subject which is now under consideration.

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